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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
This lecture summarizes recent advances in the area of numerical methods for optimal control problems, with particular emphasis on the work performed by the Aero-Astronautics Group of Rice University. The following basic problems are considered; problems with general boundary conditions, problem with nondifferential constraints, and problem with multiple subarcs.

First-order algorithms are reviewed, in particular, the sequential ordinary gradient-restoration algorithm and the sequential conjugate gradient-restoration algorithm. Second-order algorithms are also reviewed, in particular, the

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FIRST-ORDER AND SECOND-ORDER NUMERICAL METHODS
FOR OPTIMAL CONTROL PROBLEMS[1]

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Abstract

This lecture summarizes recent advances in the area of numerical methods for optimal control problems, with particular emphasis on the work performed by the Aero-Astronautics Group of Rice University. The following basic problems are considered: problem with general boundary conditions, problem with nondifferential constraints, and problem with multiple subarcs.

First-order algorithms are reviewed, in particular, the sequential ordinary gradient-restoration algorithm and the sequential conjugate gradient-restoration algorithm. Second-order algorithms are also reviewed, in particular, the modified quasilinearization algorithm. Here, the optimal initial choice of the multipliers is discussed.

Transformation techniques are presented by means of which a great variety of problems of optimal control can be reduced

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to one of the formulations presented. Specifically, the following topics are treated: time normalization, free initial state, problems with bounded control, problems with bounded state, and Chebyshev minimax problems.

Key Words

Optimal control, numerical analysis, numerical methods, computing methods, computing techniques, sequential ordinary gradient-restoration algorithm, sequential conjugate gradient-restoration algorithm, modified quasilinearization algorithm, problems with bounded control, problems with bounded state, Chebyshev minimax problems.

Introduction

In every branch of science and engineering, there exist systems which are controllable, that is, they can be made to behave in different ways depending on the will of the operator. Every time the operator of a system exerts an option (a choice in the distribution of the controls governing the system), he produces a change in the distribution of the states occupied by the system and, hence, a change in the final state. Therefore, it is natural to pose the following question: Among all the admissible options, what is the particular option which renders the system optimum? As an example, what is the option which minimizes the difference between the final value and the initial value of an arbitrarily specified function of the state

of the system? The body of knowledge covering problems of
this type is called optimal control theory. Applications
occur in every field of science and engineering, and also
economics.

It must be noted that only a minority of current technical
problems can be solved by purely analytical methods. Hence, it
is important to develop numerical techniques enabling one to
solve optimal control problems on a digital computer. These
numerical techniques can be classified into two groups: first-
order methods and second-order methods. First-order methods
(or gradient methods) are those techniques which employ at most
the first derivatives of the functions under consideration.
Second-order methods (or quasilinearization methods) are those
techniques which employ at most the second derivatives of the
functions under consideration.

Both gradient methods and quasilinearization methods re-
quire the solution of a linear, two-point boundary-value problem
(and sometimes the solution of a linear, multi-point boundary-
value problem) at every iteration. This being the case, prog-
ress in the area of numerical methods for differential equa-
tions is essential to the efficient solution of optimal control
problems on a digital computer.

Problems

For the sake of discussion, assume that the system under consideration is described by the state $x(t)$, the control $u(t)$, and the parameter π. Also, assume that the interval of integration is fixed, the initial point is given, the state is unbounded, the control is unbounded, and the parameter is unbounded. Under these conditions, we consider three basic problems, called Problem P1, Problem P2, and Problem P3 for easy identification.

Problem P1. Minimize the functional

$$I = \int_0^1 f(x,u,\pi,t)\,dt + g(x(1),\pi), \qquad (1)$$

subject to the constraints

$$\dot{x} = \phi(x,u,\pi,t), \qquad 0 \le t \le 1, \qquad (2)$$

$$x(0) = \text{given}, \qquad (3)$$

$$\psi(x(1),\pi) = 0. \qquad (4)$$

The minimization of the functional (1) must be performed with respect to the state $x(t)$, the control $u(t)$, and the parameter π, with the understanding that the time t is a scalar and x,u,π are vectors of appropriate dimensions. The functions f,g appearing in the functional (1) are scalar, and the functions ϕ,ψ appearing in the constraints (2)-(4) are vectors of appropriate dimensions.

Problem P2. Minimize the functional

$$I = \int_0^1 f(x,u,\pi,t)dt + g(x(1),\pi), \qquad (5)$$

subject to the constraints

$$\dot{x} = \phi(x,u,\pi,t), \qquad 0 \le t \le 1, \qquad (6)$$

$$S(x,u,\pi,t) = 0, \qquad 0 \le t \le 1, \qquad (7)$$

$$x(0) = given, \qquad (8)$$

$$\psi(x(1),\pi) = 0. \qquad (9)$$

Clearly, Problem P2 differs from Problem P1 because of the presence of the nondifferential constraint (7), to be satisfied everywhere along the interval of integration. The function S appearing in Eq. (7) is a vector of appropriate dimension.

Problem P3. This is a modification of Problem P2 which arises whenever the trajectories of the system include several subarcs, with the following provision: the analytical form of the functions f,ϕ,S might change from one subarc to another.

Algorithms

Over the past decade, the Aero-Astronautics Group of Rice University has successfully developed several algorithms for solving Problems P1, P2, P3 on a digital computer. Notable among these algorithms are the sequential ordinary gradient-

restoration algorithm (SOGRA, Refs. 1-4), the sequential
conjugate gradient-restoration algorithm (SCGRA, Refs. 5-8),
and the modified-quasilinearization algorithm (MQA, Refs. 9-12).
Both SOGRA and SCGRA are first-order algorithms which have a
cyclical construction. Each cycle includes a gradient phase
(one iteration) and a restoration phase (several iterations)
and is constructed in such a way that the decrease of the fun-
ctional I is guaranteed, while the constraints are satisfied
to a predetermined tolerance. On the other hand, MQA is a
second-order algorithm which has an iterative construction.
Each iteration is constructed in such a way that the decrease
of the total error in the system (the sum of the error in the
feasibility equations and the error in the optimality condi-
tions) is guaranteed.

Each iteration of SOGRA, SCGRA, and MQA requires the sol-
ution of a linear two-point boundary-value problem or a linear
multi-point boundary-value problem. In this connection, seve-
ral techniques can be useful (method of adjoint variables,
method of complementary functions, method of particular solu-
tions, and Riccati transformation). Among these, the method
of particular solutions (Refs. 13-20) has been employed in
the solution of the LTP-BVP or LMP-BVP associated with SOGRA,
SCGRA, and MQA.

Transformation Techniques

In spite of the generality of the previous work, a wide variety of problems of optimal control exist which are not covered by the above standard schemes. Whenever a nonstandard situation arises, one has two choices: (i) to develop an ad-hoc algorithm capable of solving the nonstandard problem on a digital computer; or (ii) to develop transformation techniques such that the nonstandard problem is brought within the frames of Problems P1, P2, P3. This can be achieved through proper augmentation and redefinition of the state vector $x(t)$, the control vector $u(t)$, and the parameter vector π, as well as through proper augmentation and redefinition of the constraining relations (see, for example, Refs. 21-24).

By means of transformation techniques, the following problems of optimal control can be brought within the frames of Problems P1, P2, P3: (i) problems with variable final time; (ii) problems with arbitrary conditions on the initial state; (iii) problems with control equality constraints; (iv) problems with state equality constraints; (v) problems with state-derivative equality constraints; (vi) problems with control inequality constraints; (vii) problems with state inequality constraints; (viii) problems with state-derivative inequality constraints; and (ix) minimax problems of optimal control. Therefore, by transformation techniques, the scope and range of SOGRA, SCGRA, and MQA can be increased dramatically.

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